# Note on the Relativistic Thermodynamics of Moving Bodies

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## Abstract

We employ a novel thermodynamical argument to show that, at the macroscopic level, there is no intrinsic law of temperature transformation under Lorentz boosts. This result extends the corresponding microstatistical one of earlier works to the purely macroscopic regime and signifies that the concept of temperature as an objective entity is restricted to the description of bodies in their rest frames. The argument on which this result is based is centred on the thermal transactions between a body that moves with uniform velocity relative to a certain inertial frame and a thermometer, designed to measure its temperature, that is held at rest in that frame.

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## 1. Introduction and Discussion

Classical thermodynamics has been extended to the special relativistic regime in a number of different, logically consistent, ways, which have led to different formulae for the relationship between the temperature, $T_0$, of a body in a rest frame, $K_0$, and its temperature, $T$, in an inertial frame, $K$, that moves with velocity $v$ relative to $K_0$. To be specific, in the schemes of Planck [1] and Einstein [2], $T = T_0(1 - v^2/c^2)^{1/2}$; whereas in those of Ott [3] and Kibble [4], $T = T_0(1 - v^2/c^2)^{-1/2}$; and in those of Landsberg [5], Van Kampen [6] and Callen and Horowitz [7], $T = T_0$, i.e. temperature is a scalar invariant. The relationships between the conventions and assumptions behind these different formulae has been lucidly exposited by Van Kampen [6].

In fact, all the above works were based exclusively on relativistic extensions of the first and second laws of classical thermodynamics. A different, quantum statistical, approach was introduced by Costa and Matsas [8] and by Landsberg and Matsas [9], who investigated the action of black body radiation on a monopole that moved with uniform velocity relative to the rest frame of the radiation and played the role of a thermometer or detector. The result they obtained was that the spectrum of the radiation, as registered by this detector, was non-Planckian, and therefore that it was only in a rest frame that the radiation had a well defined temperature.

A much more general version of this result was obtained by the present author [10, 11], who showed that the coupling of a moving macroscopic quantum system, $\Sigma_0$, to a fixed finite probe, $\Sigma$, drives the latter to a terminal state that, generically, is non-thermal. This signifies that, at the microstatistical level, the concept of temperature, as measured by *any*, possibly microscopic, probe is restricted to systems in their rest frames. There remain, therefore, the open questions of whether the temperature of a moving body, as registered by macroscopic observables of a probe or thermometer, is well defined and, if so, whether it transforms, under Lorentz boosts, according to some general law.

These are the questions that we address in the present article by an argument based on the classical thermodynamics of the composite, $\Sigma_c$, of two macroscopic bodies $\Sigma$ and $\Sigma_0$, subject to the following conditions. $\Sigma_0$ is in thermal equilibrium at temperature $T_0$ in a rest frame $K_0$ and moves with uniform velocity $v$ relative to a frame $K$ in which $\Sigma$ is clamped at rest. Here again $\Sigma$ serves as a thermometer for $\Sigma_0$. We investigate whether the coupling between $\Sigma$ and $\Sigma_0$ can drive $\Sigma$ to equilibrium at a temperature $T$ that depends on $T_0$ and $v$ only: if so, $T$ would be interpreted as the temperature of the moving body $\Sigma_0$, relative to the frame $K$. In fact, we show that there is no such model-independent temperature $T$. Hence, in the purely macroscopic picture, as well as in the quantum microstatistical one of Refs. [8]-[ 11], the concept of temperature as an objective entity is limited to bodies in their rest frames.

We formulate the thermodynamic description of $\Sigma_c = (\Sigma + \Sigma_0)$ in Sec. 2, concluding that Section with the observation that its entropy can increase indefinitely and therefore that it cannot evolve into a true equilibrium state. This, however, does not preclude the possibility that $\Sigma_0$ might drive $\Sigma$ into an equilibrium state, and in Sec. 3 we investigate this possibility for a specific tractable model in which $\Sigma$ and $\Sigma_0$ interact via emission and

absorption of radiation. This model is a variant of the one constructed by Van Kampen [6] for his treatment of heterotachic processes. We show that, for this model, $\Sigma$ is indeed driven into a thermal equilibrium state, but that the resultant temperature depends on variable parameters of this system. Accordingly we conclude in Sec. 4 that, since the temperature attained by $\Sigma$ is just that of the moving body $\Sigma_0$, as measured by a fixed thermometer, there is no intrinsic law of temperature transformations under Lorentz boosts. This result extands those of [8]-[11] from the microstatistical picture to the purely macroscopic one.

## 2. The Thermodynamic Description

Let $\Sigma_0$ be a macroscopic system that moves with velocity $v$ relative to an inertial frame $K$ and that is in equilibrium at temperature $T_0$ relative to a rest frame $K_0$. In order to formulate its thermodynamics relative to $K$, we consider the situation in which it is placed in diathermic interaction with a macroscopic probe, $\Sigma$, that is clamped at rest relative to $K$. We assume that the clamp is infinitely massive, and therefore immovable, and that its action on $\Sigma$ is adiabatic. Under these conditions, there is no thermal or mechanical exchange of energy, relative to $K$, between $\Sigma$ and the clamp. Further, we assume that the systems $\Sigma$ and $\Sigma_0$ are spatially separated, so that they do not exchange energy by mechanical means.

The transactions between $\Sigma_0$ and $\Sigma$ constitute a heterotachic process, as defined by Van Kampen [6], but with the crucial constraint that the momentum of $\Sigma$, relative to $K$, is held at the value zero. In this process, the energy relative to $K$ of the composite $\Sigma_c = (\Sigma + \Sigma_0)$ is conserved, but its momentum is not: any momentum received by $\Sigma$ is immediately discharged into the immovable clamp.

We assume that, although both $\Sigma$ and $\Sigma_0$ are macroscopic, the former is of much smaller size than the latter in that, if $\Omega$ and $\Omega_0$ are dimensionless extensivity parameters (e.g. particles numbers) that provide measures of their respective sizes, then $\Omega_0 >> \Omega >> 1$. In order to sharpen our formulation, we take $\Sigma_0$ to be an infinite system, as in [10, 11], so that $\Omega_0 = \infty$. Thus, $\Sigma_0$ serves as a thermal reservoir whose temperature and pressure remain constant during its transactions with $\Sigma$.

We assume, for simplicity, that the energy $E$ and volume $V$ of $\Sigma$, relative to the rest frame $K$, constitute a complete set of its extensive thermodynamical variables*. In fact, $V$ is merely constant during the transactions between this system and $\Sigma_0$ since, as stipulated above, no mechanical work is done on it relative to its rest frame. As for $\Sigma_0$, we assume that its temperature $T_0$ and pressure $\Pi_0$, relative to $K_0$, together with its velocity $v$ relative to $K$, constitute a complete set of its intensive thermodynamic control variables. Finite changes from the equilibrium state of this system are given by increments $E_0$ and $P_0$ of its energy and momentum, respectively, relative to $K_0$. Hence, by Lorentz transformation, the increment in its energy relative to $K$ is $(1 - v^2/c^2)^{-1/2}(E_0 + v.P_0)$ and therefore the

* A general quantum statistical characterisation of a complete set of extensive thermodynamical variables is provided in [12, Sec. 6.4].

conservation of energy condition for $\Sigma_c$, relative to $K$, is .

$$E + \gamma(E_0 + v.P_0) = const., \tag{2.1}$$

where

$$\gamma = (1 - v^2/c^2)^{-1/2}. \tag{2.2}$$

Note that it would be wrong to assume energy conservation relative to $K_0$, since energy in this frame is a linear combination of energy and momentum in $K$, and the clamping condition destroys the conservation of momentum of $\Sigma_c$ relative to the latter frame.

The entropy of $\Sigma$ is a function $S$ of $E$ and $V$, which is jointly concave in its arguments [13, Sec. 1.10], and its value is Lorentz invariant [6; 14, Sec. 46]. The temperature $T$ of $\Sigma$ is related to $S$ by the standard formula

$$T^{-1} = \frac{\partial S(E,V)}{\partial E}. \tag{2.3}$$

Since $K_0$ is a rest frame for $\Sigma_0$, the incremental entropy of this system, due to modification of its equilibrium state by changes $E_0$ and $P_0$ of its energy and momentum relative to this frame, is simply

$$S_0(E_0) = T_0^{-1} E_0. \tag{2.4}$$

The total entropy of the composite $\Sigma_c$, as measured relative to the specified equilibrium state of $\Sigma_0$, is just the sum of those of $\Sigma$ and $\Sigma_0$, which, by Eqs. (2.1) and (2.4), is equal to $S(E,V) - T_0^{-1}(\gamma^{-1}E + v.P_0)$, plus a constant. Hence, defining

$$\tilde{T} = \gamma T_0 \tag{2.5}$$

and

$$\tilde{S}(E,V) = S(E,V) - \tilde{T}^{-1}E, \tag{2.6}$$

the entropy of $\Sigma_c$ is

$$S_c(E,V;P_0) = \tilde{S}(E,V) - T_0^{-1} v.P_0 + const.. \tag{2.7}$$

We now note that it follows from Eq. (2.6) and the concavity of $S$ that $\tilde{S}$ is maximised at the value of $E$ for which $\partial S(E,V)/\partial E = \tilde{T}^{-1}$ and that the resultant value of $\tilde{S}$ is the finite quantity given by $-\tilde{T}^{-1}$ times the Helmholtz free energy of $\Sigma$ at temperature $\tilde{T}$ and volume $V$ [13, Sec.5.3]. On the other hand, the second term on the r.h.s. of Eq. (2.7) increases indefinitely with the modulus of $P_0$ when the direction of this excess momentum opposes that of $v$. Hence, $S_c$ has no finite upper bound and so we reach the following conclusion.

(I) *Under the prescribed conditions, the composite system $\Sigma_c$ does not support any equilibrium state, as defined by the maximum entropy condition.*

Of course this does not rule out the possibility that $\Sigma$ might be driven into a thermal state, with well defined temperature, as a result of its interaction with $\Sigma_0$. In the following Section, we shall show that this possibility is realised by a tractable model, but that the resultant temperature varies with the parameters of the model.

## 3. The Radiative Transfer Model

The model presented here is a variant of Van Kampen's [6] system of two bodies that interact by radiation through a small hole in a metallic sheet placed between them. In the present context, these systems are the above described ones $\Sigma$ and $\Sigma_0$. We assume that their respective boundaries facing the sheet are plane surfaces, $F$ and $F_0$, that are parallel both to it and to the velocity $v$. We assume that the sheet and the face $F_0$ are unbounded and that the sheet is at rest relative to $K$. Further, we assume that the hole is in the part of the sheet given by the orthogonal projection of $F$ onto it and that both the linear span of the hole and its distance from $F$* are negligibly small by comparison with its distance from the boundary of that face.

The modifications of Van Kampen's model that we introduce here are the following.

• Only $\Sigma_0$, but not $\Sigma$, is a black body. We denote by $A(\omega)$ the absorption coefficient of $\Sigma$ for radiation of frequency $\omega$. By Kirchoff's law [15, Sec. 60], it is also the emission coefficient of this system, and it necessarily lies in the interval [0,1].

• $\Sigma$ is clamped at rest in $K$.

• No radiation emanating from $\Sigma_0$ falls on the clamp: this can be achieved by placing $\Sigma$ between the hole and the clamp.

**3.1. The Energy Exchanges.** Our treatment of the transactions between $\Sigma$ and $\Sigma_0$ will be basd on a calculation of the increment in the energy, $\Delta E$, of $\Sigma$ relative to $K$ in time $\Delta t$. Evidently this may be expressed in the form

$$\Delta E = \Delta E_2 - \Delta E_1, \tag{3.1}$$

where $\Delta E_1$ (resp. $\Delta E_2$) is the energy transferred from $\Sigma$ to $\Sigma_0$ (resp. $\Sigma_0$ to $\Sigma$) in that time. These energy transfers are achieved by leaks of the radiations emanating from $\Sigma$ and $\Sigma_0$ through the hole in the metallic sheet. Since both the linear span of the hole and its distance from $F$ are negligible by comparison with its distance from the boundary of $F$, we may assume, for the purpose of calculating $\Delta E$, that the face $F$, as well as $F_0$, is infinitely extended. We denote by $\Gamma$ (resp. $\Gamma_0$) the region bounded by $F$ (resp. $F_0$) and the sheet. Thus $\Gamma$ and $\Gamma_0$ are are filled with the thermal radiation emanating from $\Sigma$ and $\Sigma_0$, respectively, as modified by the leakages through the hole.

In order to calculate $\Delta E_1$, we first note that the energy density of the radiation in $\Gamma$ that lies in the infinitesimal frequency range $[\omega, \omega + d\omega]$ and whose direction lies in a solid

* The distance of the hole from $F$ has to be so small in order to suppress end effects at the boundary of that surface.

angle $d\Omega$ is $A(\omega)\omega^3[\exp(\hbar w/kT)-1]^{-1}d\omega d\Omega$, times a universal constant. Hence, denoting the area of the hole by $\Delta a$, the energy transferred by this pencil of radiation from $\Gamma$ to $\Gamma_0$ in time $\Delta t$ is

$$C\Delta a\Delta t A(\omega)\omega^3[\exp(\hbar\omega/kT)-1]^{-1}\cos(\psi)d\omega d\Omega,$$

where $C$ is a universal constant and $\psi$ is the angle between the pencil and the outward drawn normal to the sheet. It is convenient to express $d\Omega$ and $\cos(\psi)$ in terms of spherical polar coordinates $\theta$ ($\in[0,\pi]$) and $\phi$ ($\in[-\pi/2,\pi/2]$), where the former is the angle between the pencil and the direction of $v$ and the latter is the azimuthal angle of rotation of the pencil about the line of $v$. Specifically,

$$d\Omega=\sin(\theta)d\theta d\phi \text{ and } \cos(\psi)=\sin(\theta)\cos(\phi)$$

and therefore the above expression for the energy transferred across the hole from $\Gamma$ may be re-expressed as

$$C\Delta a\Delta t A(\omega)\omega^3[\exp(\hbar\omega/kT)-1]^{-1}\sin^2(\theta)\cos(\phi)d\omega d\theta d\phi, \tag{3.2}$$

Since $\Sigma_0$ is a black body, the total energy $\Delta E_1$, relative to $K$, that is transferred from $\Sigma$ to $\Sigma_0$ in time $\Delta t$ is obtained by integration of this quantity over the ranges $[0,\infty]$ for $\omega$, $[0,\pi]$ for $\theta$ and $[-\pi/2,\pi/2]$ for $\phi$. Thus

$$\Delta E_1=C\Phi(T)\Delta a\Delta t, \tag{3.3}$$

where

$$\Phi(T)=\pi\int_0^\infty d\omega A(\omega)\omega^3[\exp(\hbar\omega/kT)-1]^{-1}. \tag{3.4}$$

Here there is the tacit mathematical assumption that the function $A$ is measurable: otherwise the integral in Eq. (3.4) would not be well defined. However, from the physical standpoint, this assumption is very mild, as it is satisfied if $A$ is piecewise continuous. It follows from Eq. (3.4) that $\Phi(T)$ is a continuous and monotonically increasing function of $T$ whose range is $[0,\infty]$.

The calculation of $\Delta E_2$ proceeds along similar lines, with modifications due to the motion of $\Sigma_0$ relative to $K$. To effect this calculation we first note that the radiation emanating from the black body $\Sigma_0$ is Planckian, and therefore isotropic, relative to $K_0$.We then define $\omega_0$, $\theta_0$ and $\phi_0$ to be the natural counterparts of $\omega$, $\theta$ and $\phi$, respectively, for the description of $\Sigma_0$ relative to $K_0$, and we denote by $\mathcal{P}_0$ the pencil of radiation emanating from $\Sigma_0$ for which these variables lie in the infinitesimal ranges $[\omega_0,\omega_0+d\omega_0]$, $[\theta_0,\theta_0+d\theta_0]$ and $[\phi_0,\phi_0+d\phi_0]$. We then note that $\Delta a\Delta t$ is Lorentz invariant, i.e. it is equal to the product of the counterparts $\Delta a_0$ and $\Delta t_0$ of $\Delta a$ and $\Delta t$ relative to the frame $K_0$. It now follows by simple analogy with the derivation of (3.2) that the energy, relative to $K_0$, that is transferred by this pencil through the hole from the in time $\Delta t_0$ is given by the canonical analogue of the expression (3.2), but with the term $A(\omega)$ omitted, since $\Sigma_0$ is a black body. Hence, in view of the Lorentz invariance of $\Delta a\Delta t$, the energy relative to $K_0$ transmitted by the pencil $\mathcal{P}_0$ through the hole in time $\Delta t_0$ is

$$C\Delta a\Delta t\omega_0^3[\exp(\hbar\omega_0/kT_0)-1]^{-1}\sin^2(\theta_0)\cos(\phi_0)d\omega_0 d\theta_0 d\phi_0.$$

Correspondingly, the component parallel to $v$ of the momentum of $\mathcal{P}_0$, relative to $K_0$, that is transferred from $\Gamma_0$ to $\Gamma$ in time $\Delta t_0$ is just $c^{-1}\cos(\theta_0)$ times this quantity. Hence, by Lorentz transformation, the energy of this pencil, relative to $K$, that is transferred to $\Sigma$ in time $\Delta t$ is

$$C\gamma\Delta a\Delta t\omega_0^3[\exp(\hbar\omega_0/kT_0)-1]^{-1}\big(1+(|v|/c)\cos(\theta_0)\big)\sin^2(\theta_0)\cos(\phi_0)d\omega_0 d\theta_0 d\phi_0. \tag{3.5}$$

Moreover, in view of the relativistic Doppler effect [14, Sec. 6], the frequency of this radiative pencil, relative to $K$, is

$$\omega=\gamma\big(1+(|v|/c)\cos(\theta_0)\big)\omega_0. \tag{3.6}$$

Therefore, as viewed in $K$, the energy transferred by the pencil $\mathcal{P}_0$ from $\Sigma_0$ to $\Sigma$ in time $\Delta t$ is just $\gamma$ times the expression (3.5), but with $\omega_0$ replaced by $\gamma^{-1}\big(1+(|v|/c)\cos(\theta_0)\big)^{-1}\omega$. Moreover, the resultant energy absorbed by $\Sigma$ from the pencil is just the absorption coefficient $A(\omega)$ times this quantity. The total energy $\Delta E_2$ absorbed by $\Sigma$ in time $\Delta t$ is then obtained by integration and takes the form

$$\Delta E_2=C\Phi_0(T_0)\Delta a\Delta t, \tag{3.7}$$

where

$$\Phi_0(T_0)=2\gamma^{-3}\int_0^\infty d\omega\int_0^\pi d\theta_0 A(\omega)\omega^3\sin^2(\theta_0)\times$$
$$\big(1+(|v|/c)\cos(\theta_0)\big)^{-3}\Big[\exp\Big((\hbar\omega/\gamma kT)\big(1+(|v|/c)\cos(\theta_0)\big)^{-1}\Big)-1\Big]^{-1}. \tag{3.8}$$

It follows immediately from this formula that $\Phi_0$ is a continuous, monotonically increasing function of $T_0$ whose range is $[0,\infty)$.

We now infer from Eqs. (3.1), (3.3) and (3.7) that the net energy increment in the energy of $\Sigma$, relative to $K$, in time $\Delta t$ is

$$\Delta E=C[\Phi_0(T_0)-\Phi(T)]\Delta a\Delta t. \tag{3.9}$$

Hence, passing to the limit $\Delta t\to 0$, the rate of change of the energy $E$ of $\Sigma$ is

$$\frac{dE}{dt}=C[\Phi_0(T_0)-\Phi(T)]\Delta a. \tag{3.10}$$

**3.2. Evolution to the Equilibrium Temperature of $\Sigma$.** Since the functions $\Phi$ and $\Phi_0$ are continuous and monotonically increasing, with range $[0,\ \infty)$, it follows from Eq. (3.10) that there is precisely one value, $\overline{T}$, of $T$ for which $E$ is stationary. Thus $\overline{T}$ is determined by the equation

$$\Phi(\overline{T})=\Phi_0(T_0). \tag{3.11}$$

Moreover, since $\Phi_0$, as well as $\Phi$, increases monotonically and continuously with its argument, this formula implies that $\overline{T}$ is an increasing function of $T_0$.

In order to show that the temperature of $\Sigma$ evolves irreversibly to the value $\overline{T}$, we introduce the free energy function

$$F(E,V)=E-\overline{T}S(E,V) \tag{3.12}$$

and infer from Eqs. (2.3) and (3.10) that

$$\frac{d}{dt}F(E,V)=C[1-\overline{T}/T][\Phi_0(T_0)-\Phi(T)]\Delta a$$

and consequently, by Eq. (3.11), that

$$\frac{d}{dt}F(E,V)=C[1-\overline{T}/T][\Phi(\overline{T})-\Phi(T)]\Delta a. \tag{3.13}$$

Since $\Phi$ is a continuous monotonically increasing function of temperature it follows immediately from this equation that $dF/dt$ is negative except at $T=\overline{T}$, where it is zero. This leads us to the following result.

(II) *$F$ serves as a Lyapounov function whose monotonic decrease with time ensures that the temperature of $\Sigma$ evolves irreversibly to a stable terminal value $\overline{T}$, which is the temperature of the moving system $\Sigma_0$, as registered by the thermometer fixed in $K$. Moreover, as noted following Eq. (3.9), this temperature is an increasing function of $T_0$.*

**3.3. Dependence of $\overline{T}$ on the Parameters of the Model.** We now remark that, by Eqs. (3.4) and (3.8), the functions $\Phi$ and $\Phi_0$ depend on the form of the absorption coefficient $A(\omega)$, and therefore, by Eq. (3.11), so too does the temperature $\overline{T}$. In order to establish that this dependence is non-trivial, we consider the case where $A(\omega)$ is unity when $\omega$ lies in a narrow interval $[f,f+\Delta f]$ and is otherwise zero. In this case, it follows from Eqs. (3.4), (3.8) and (3.11) that

$$[\exp(\hbar f/k\overline{T})-1]^{-1}=2\pi^{-1}\gamma^{-3}\int_0^\pi d\theta_0\sin^2(\theta_0)\times$$

$$\big(1+(|v|/c)\cos(\theta_0)\big)^{-3}\big[\exp\big((\hbar f/\gamma kT_0)\big(1+(|v|/c)\cos(\theta_0)\big)^{-1}\big)-1\big]^{-1}. \tag{3.14}$$

In order to establish that $\overline{T}$ depends non-trivially on the frequency $f$, i.e. that it is not just a constant, we show $\overline{T}$ tends to different limits as $f$ tends to zero and infinity. Thus, in the case of small $f$, we may approximate the quantities in the square brackets on the left and right hand sides of Eq. (3.14) by the exponents occurring there. Thus we find that

$$\overline{T}\to 2\pi^{-1}\gamma^{-2}T_0\int_0^\pi d\theta_0\sin^2(\theta_0)\big(1+(|v|/c)\cos(\theta_0)\big)^{-2} \text{ as } f\to 0. \tag{3.15}$$

On the other hand, for large $f$, we may discount the terms $-1$ in the square brackets on both sides of Eq. (3.14), thereby obtaining the formula

$$\exp(-\hbar f/k\overline{T})=2\pi^{-1}\gamma^{-3}\times$$

$$\int_0^\pi d\theta_0 \sin^2(\theta_0)\big(1+(|v|/c)\cos(\theta_0)\big)^{-3}\exp\big(-(\hbar f/\gamma kT_0)\big(1+(|v|/c)\cos(\theta_0)\big)^{-1}\big).$$

For large $f$, the r.h.s. of this equation is dominated by the exponential term occurring therein and its logarithm reduces to the maximum value of the exponent for $\theta_0 \in [0,\pi]$. Hence, using Eq. (2.2), we find that

$$\overline{T} \to T_0\Big(\frac{1+|v|/c}{1-|v|/c}\Big)^{1/2} \text{ as } f\to\infty. \tag{3.16}$$

This limit is evidently different from that of Eq. (3.15), since it follows easily from Eq. (2.2) that the latter limit is equal to $T_0\big(1+O(v^2/c^2)\big)$. Hence the temperature $\overline{T}$ must be a nontrivial, i.e. non-constant, function of $f$. It follows that this temperature depends on the parameter of the model and therefore we arrive at the following general conclusion.

(III) *According to the purely macroscopic picture, there is no intrinsic law of temperature transformations under Lorentz boosts.*

## 4. Conclusion

Our essential results are encapsulated by the assertions (I) of Sec. 2 and (II) and (III) of Sec. 3. The first of these is that, under the prescribed conditions, the composite of $(\Sigma+\Sigma_0)$ cannot evolve to an equilibrium state, as given by the maximum entropy condition. However, as in the case of Sec. 3, where these systems interact via radiative transfer, their coupling can drive $\Sigma$ into an equilibrium state whose temperature $\overline{T}$ varies not only with $T_0$ but also with the parameters of the thermometer $\Sigma$. From this we conclude that in the purely macroscopic picture, as in the microstatistical one of [8]-[11], there is no intrinsic law of temperature transformation under Lorentz boosts.

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In fact, all the above works were based exclusively on relativistic extensions of the first and second laws of classical thermodynamics. A different, quantum statistical, approach was introduced by Costa and Matsas [8] and by Landsberg and Matsas [9], who investigated the action of black body radiation on a monopole that moved with uniform velocity relative to the rest frame of the radiation and played the role of a thermometer or detector. The result they obtained was that the spectrum of the radiation, as registered by this detector, was non-Planckian, and therefore that it was only in a rest frame that the radiation had a well defined temperature.

A much more general version of this result was obtained by the present author [10, 11], who showed that the coupling of a moving macroscopic quantum system, $\Sigma_0$, to a fixed finite probe, $\Sigma$, drives the latter to a terminal state that, generically, is non-thermal. This signifies that, at the microstatistical level, the concept of temperature, as measured by *any*, possibly microscopic, probe is restricted to systems in their rest frames. There remain, therefore, the open questions of whether the temperature of a moving body, as registered by macroscopic observables of a probe or thermometer, is well defined and, if so, whether it transforms, under Lorentz boosts, according to some general law.

These are the questions that we address in the present article by an argument based on the classical thermodynamics of the composite, $\Sigma_c$, of two macroscopic bodies $\Sigma$ and $\Sigma_0$, subject to the following conditions. $\Sigma_0$ is in thermal equilibrium at temperature $T_0$ in a rest frame $K_0$ and moves with uniform velocity $v$ relative to a frame $K$ in which $\Sigma$ is clamped at rest. Here again $\Sigma$ serves as a thermometer for $\Sigma_0$. We investigate whether the coupling between $\Sigma$ and $\Sigma_0$ can drive $\Sigma$ to equilibrium at a temperature $T$ that depends on $T_0$ and $v$ only: if so, $T$ would be interpreted as the temperature of the moving body $\Sigma_0$, relative to the frame $K$. In fact, we show that there is no such model-independent temperature $T$. Hence, in the purely macroscopic picture, as well as in the quantum microstatistical one of Refs. [8]-[ 11], the concept of temperature as an objective entity is limited to bodies in their rest frames.

We formulate the thermodynamic description of $\Sigma_c = (\Sigma + \Sigma_0)$ in Sec. 2, concluding that Section with the observation that its entropy can increase indefinitely and therefore that it cannot evolve into a true equilibrium state. This, however, does not preclude the possibility that $\Sigma_0$ might drive $\Sigma$ into an equilibrium state, and in Sec. 3 we investigate this possibility for a specific tractable model in which $\Sigma$ and $\Sigma_0$ interact via emission and

absorption of radiation. This model is a variant of the one constructed by Van Kampen [6] for his treatment of heterotachic processes. We show that, for this model, $\Sigma$ is indeed driven into a thermal equilibrium state, but that the resultant temperature depends on variable parameters of this system. Accordingly we conclude in Sec. 4 that, since the temperature attained by $\Sigma$ is just that of the moving body $\Sigma_0$, as measured by a fixed thermometer, there is no intrinsic law of temperature transformations under Lorentz boosts. This result extands those of [8]-[11] from the microstatistical picture to the purely macroscopic one.

## 2. The Thermodynamic Description

Let $\Sigma_0$ be a macroscopic system that moves with velocity $v$ relative to an inertial frame $K$ and that is in equilibrium at temperature $T_0$ relative to a rest frame $K_0$. In order to formulate its thermodynamics relative to $K$, we consider the situation in which it is placed in diathermic interaction with a macroscopic probe, $\Sigma$, that is clamped at rest relative to $K$. We assume that the clamp is infinitely massive, and therefore immovable, and that its action on $\Sigma$ is adiabatic. Under these conditions, there is no thermal or mechanical exchange of energy, relative to $K$, between $\Sigma$ and the clamp. Further, we assume that the systems $\Sigma$ and $\Sigma_0$ are spatially separated, so that they do not exchange energy by mechanical means.

The transactions between $\Sigma_0$ and $\Sigma$ constitute a heterotachic process, as defined by Van Kampen [6], but with the crucial constraint that the momentum of $\Sigma$, relative to $K$, is held at the value zero. In this process, the energy relative to $K$ of the composite $\Sigma_c = (\Sigma + \Sigma_0)$ is conserved, but its momentum is not: any momentum received by $\Sigma$ is immediately discharged into the immovable clamp.

We assume that, although both $\Sigma$ and $\Sigma_0$ are macroscopic, the former is of much smaller size than the latter in that, if $\Omega$ and $\Omega_0$ are dimensionless extensivity parameters (e.g. particles numbers) that provide measures of their respective sizes, then $\Omega_0 >> \Omega >> 1$. In order to sharpen our formulation, we take $\Sigma_0$ to be an infinite system, as in [10, 11], so that $\Omega_0 = \infty$. Thus, $\Sigma_0$ serves as a thermal reservoir whose temperature and pressure remain constant during its transactions with $\Sigma$.

We assume, for simplicity, that the energy $E$ and volume $V$ of $\Sigma$, relative to the rest frame $K$, constitute a complete set of its extensive thermodynamical variables*. In fact, $V$ is merely constant during the transactions between this system and $\Sigma_0$ since, as stipulated above, no mechanical work is done on it relative to its rest frame. As for $\Sigma_0$, we assume that its temperature $T_0$ and pressure $\Pi_0$, relative to $K_0$, together with its velocity $v$ relative to $K$, constitute a complete set of its intensive thermodynamic control variables. Finite changes from the equilibrium state of this system are given by increments $E_0$ and $P_0$ of its energy and momentum, respectively, relative to $K_0$. Hence, by Lorentz transformation, the increment in its energy relative to $K$ is $(1 - v^2/c^2)^{-1/2}(E_0 + v.P_0)$ and therefore the

* A general quantum statistical characterisation of a complete set of extensive thermodynamical variables is provided in [12, Sec. 6.4].

conservation of energy condition for $\Sigma_c$, relative to $K$, is .

$$E + \gamma(E_0 + v.P_0) = const., \tag{2.1}$$

where

$$\gamma = (1 - v^2/c^2)^{-1/2}. \tag{2.2}$$

Note that it would be wrong to assume energy conservation relative to $K_0$, since energy in this frame is a linear combination of energy and momentum in $K$, and the clamping condition destroys the conservation of momentum of $\Sigma_c$ relative to the latter frame.

The entropy of $\Sigma$ is a function $S$ of $E$ and $V$, which is jointly concave in its arguments [13, Sec. 1.10], and its value is Lorentz invariant [6; 14, Sec. 46]. The temperature $T$ of $\Sigma$ is related to $S$ by the standard formula

$$T^{-1} = \frac{\partial S(E,V)}{\partial E}. \tag{2.3}$$

Since $K_0$ is a rest frame for $\Sigma_0$, the incremental entropy of this system, due to modification of its equilibrium state by changes $E_0$ and $P_0$ of its energy and momentum relative to this frame, is simply

$$S_0(E_0) = T_0^{-1} E_0. \tag{2.4}$$

The total entropy of the composite $\Sigma_c$, as measured relative to the specified equilibrium state of $\Sigma_0$, is just the sum of those of $\Sigma$ and $\Sigma_0$, which, by Eqs. (2.1) and (2.4), is equal to $S(E,V) - T_0^{-1}(\gamma^{-1}E + v.P_0)$, plus a constant. Hence, defining

$$\tilde{T} = \gamma T_0 \tag{2.5}$$

and

$$\tilde{S}(E,V) = S(E,V) - \tilde{T}^{-1}E, \tag{2.6}$$

the entropy of $\Sigma_c$ is

$$S_c(E,V;P_0) = \tilde{S}(E,V) - T_0^{-1} v.P_0 + const.. \tag{2.7}$$

We now note that it follows from Eq. (2.6) and the concavity of $S$ that $\tilde{S}$ is maximised at the value of $E$ for which $\partial S(E,V)/\partial E = \tilde{T}^{-1}$ and that the resultant value of $\tilde{S}$ is the finite quantity given by $-\tilde{T}^{-1}$ times the Helmholtz free energy of $\Sigma$ at temperature $\tilde{T}$ and volume $V$ [13, Sec.5.3]. On the other hand, the second term on the r.h.s. of Eq. (2.7) increases indefinitely with the modulus of $P_0$ when the direction of this excess momentum opposes that of $v$. Hence, $S_c$ has no finite upper bound and so we reach the following conclusion.

(I) *Under the prescribed conditions, the composite system $\Sigma_c$ does not support any equilibrium state, as defined by the maximum entropy condition.*

Of course this does not rule out the possibility that $\Sigma$ might be driven into a thermal state, with well defined temperature, as a result of its interaction with $\Sigma_0$. In the following Section, we shall show that this possibility is realised by a tractable model, but that the resultant temperature varies with the parameters of the model.

## 3. The Radiative Transfer Model

The model presented here is a variant of Van Kampen's [6] system of two bodies that interact by radiation through a small hole in a metallic sheet placed between them. In the present context, these systems are the above described ones $\Sigma$ and $\Sigma_0$. We assume that their respective boundaries facing the sheet are plane surfaces, $F$ and $F_0$, that are parallel both to it and to the velocity $v$. We assume that the sheet and the face $F_0$ are unbounded and that the sheet is at rest relative to $K$. Further, we assume that the hole is in the part of the sheet given by the orthogonal projection of $F$ onto it and that both the linear span of the hole and its distance from $F$ are negligibly small* by comparison with its distance from the boundary of that face.

The modifications of Van Kampen's model that we introduce here are the following.

- Only $\Sigma_0$, but not $\Sigma$, is a black body. We denote by $A(\omega)$ the absorption coefficient of $\Sigma$ for radiation of frequency $\omega$. By Kirchoff's law [15, Sec. 60], it is also the emission coefficient of this system, and it necessarily lies in the interval [0,1].
- $\Sigma$ is clamped at rest in $K$.
- No radiation emanating from $\Sigma_0$ falls on the clamp: this can be achieved by placing $\Sigma$ between the hole and the clamp.

**3.1. The Energy Exchanges.** Our treatment of the transactions between $\Sigma$ and $\Sigma_0$ will be basd on a calculation of the increment in the energy, $\Delta E$, of $\Sigma$ relative to $K$ in time $\Delta t$. Evidently this may be expressed in the form

$$\Delta E = \Delta E_2 - \Delta E_1, \tag{3.1}$$

where $\Delta E_1$ (resp. $\Delta E_2$) is the energy transferred from $\Sigma$ to $\Sigma_0$ (resp. $\Sigma_0$ to $\Sigma$) in that time. These energy transfers are achieved by leaks of the radiations emanating from $\Sigma$ and $\Sigma_0$ through the hole in the metallic sheet. Since both the linear span of the hole and its distance from $F$ are negligible by comparison with its distance from the boundary of $F$, we may assume, for the purpose of calculating $\Delta E$, that the face $F$, as well as $F_0$, is infinitely extended. We denote by $\Gamma$ (resp. $\Gamma_0$) the region bounded by $F$ (resp. $F_0$) and the sheet. Thus $\Gamma$ and $\Gamma_0$ are are filled with the thermal radiation emanating from $\Sigma$ and $\Sigma_0$, respectively, as modified by the leakages through the hole.

In order to calculate $\Delta E_1$, we first note that the energy density of the pencil of radiation in $\Gamma$ that lies in the infinitesimal frequency range $[\omega, \omega + d\omega]$ and whose direction

* The distance of the hole from $F$ has to be so small in order to suppress end effects at the boundary of that surface.

lies in a solid angle $d\Omega$ is $A(\omega)\omega^3[\exp(\hbar w/kT)-1]^{-1}d\omega d\Omega$, times a universal constant. Hence, denoting the area of the hole by $\Delta a$, the energy transferred by this pencil of radiation from $\Gamma$ to $\Gamma_0$ in time $\Delta t$ is

$$C\Delta a\Delta t A(\omega)\omega^3[\exp(\hbar\omega/kT)-1]^{-1}\cos(\psi)d\omega d\Omega,$$

where $C$ is a universal constant and $\psi$ is the angle between the pencil and the outward drawn normal to the sheet. It is convenient to express $d\Omega$ and $\cos(\psi)$ in terms of spherical polar coordinates $\theta$ ($\in[0,\pi]$) and $\phi$ ($\in[-\pi/2,\pi/2]$), where the former is the angle between the pencil and the direction of $v$ and the latter is the azimuthal angle of rotation of the pencil about the line of $v$. Specifically,

$$d\Omega=\sin(\theta)d\theta d\phi \text{ and } \cos(\psi)=\sin(\theta)\cos(\phi)$$

and therefore the above expression for the energy transferred across the hole from $\Gamma$ may be re-expressed as

$$C\Delta a\Delta t A(\omega)\omega^3[\exp(\hbar\omega/kT)-1]^{-1}\sin^2(\theta)\cos(\phi)d\omega d\theta d\phi, \tag{3.2}$$

Since $\Sigma_0$ is a black body, the total energy $\Delta E_1$, relative to $K$, that is transferred from $\Sigma$ to $\Sigma_0$ in time $\Delta t$ is obtained by integration of this quantity over the ranges $[0,\infty]$ for $\omega$, $[0,\pi]$ for $\theta$ and $[-\pi/2,\pi/2]$ for $\phi$. Thus

$$\Delta E_1=C\Phi(T)\Delta a\Delta t, \tag{3.3}$$

where

$$\Phi(T)=\pi\int_0^\infty d\omega A(\omega)\omega^3[\exp(\hbar\omega/kT)-1]^{-1}. \tag{3.4}$$

Here there is the tacit mathematical assumption that the function $A$ is measurable: otherwise the integral in Eq. (3.4) would not be well defined. However, from the physical standpoint, this assumption is very mild, as it is satisfied if the function $A$ is piecewise continuous. It follows from Eq. (3.4) that $\Phi(T)$ is a continuous and monotonically increasing function of $T$ whose range is $[0,\infty]$.

The calculation of $\Delta E_2$ proceeds along similar lines, with modifications due to the motion of $\Sigma_0$ relative to $K$. To effect this calculation we first note that the radiation emanating from the black body $\Sigma_0$ is Planckian, and therefore isotropic, relative to $K_0$.We then define $\omega_0$, $\theta_0$ and $\phi_0$ to be the natural counterparts of $\omega$, $\theta$ and $\phi$, respectively, for the description of $\Sigma_0$ relative to $K_0$, and we denote by $\mathcal{P}_0$ the pencil of radiation emanating from $\Sigma_0$ for which these variables lie in the infinitesimal ranges $[\omega_0,\omega_0+d\omega_0]$, $[\theta_0,\theta_0+d\theta_0]$ and $[\phi_0,\phi_0+d\phi_0]$. We then note that $\Delta a\Delta t$ is Lorentz invariant, i.e. it is equal to the product of the counterparts $\Delta a_0$ and $\Delta t_0$ of $\Delta a$ and $\Delta t$ relative to the frame $K_0$. It now follows by simple analogy with the derivation of (3.2) that the energy, relative to $K_0$, that is transferred by this pencil through the hole from $\Gamma_0$ to $\Gamma$ in time $\Delta t_0$ is given by the canonical analogue of the expression (3.2), but with the term $A(\omega)$ omitted, since $\Sigma_0$ is a

black body. Hence, in view of the Lorentz invariance of $\Delta a \Delta t$, the energy relative to $K_0$ transmitted by the pencil $\mathcal{P}_0$ through the hole in time $\Delta t_0$ is

$$C \Delta a \Delta t \omega_0^3 [\exp(\hbar\omega_0/kT_0) - 1]^{-1} \sin^2(\theta_0) \cos(\phi_0) d\omega_0 d\theta_0 d\phi_0.$$

Correspondingly, the component parallel to $v$ of the momentum of $\mathcal{P}_0$, relative to $K_0$, that is transferred from $\Gamma_0$ to $\Gamma$ in time $\Delta t_0$ is just $c^{-1}\cos(\theta_0)$ times this quantity. Hence, by Lorentz transformation, the energy of this pencil, relative to $K$, that is transferred to $\Sigma$ in time $\Delta t$ is

$$C\gamma \Delta a \Delta t \omega_0^3 [\exp(\hbar\omega_0/kT_0) - 1]^{-1} \big(1 + (|v|/c)\cos(\theta_0)\big) \sin^2(\theta_0) \cos(\phi_0) d\omega_0 d\theta_0 d\phi_0. \tag{3.5}$$

Moreover, in view of the relativistic Doppler effect [14, Sec. 6], the frequency of this radiative pencil, relative to $K$, is

$$\omega = \gamma \big(1 + (|v|/c)\cos(\theta_0)\big)\omega_0. \tag{3.6}$$

Therefore, as viewed in $K$, the energy transferred by the pencil $\mathcal{P}_0$ from $\Sigma_0$ to $\Sigma$ in time $\Delta t$ is just $\gamma$ times the expression (3.5), but with $\omega_0$ replaced by $\gamma^{-1}\big(1+(|v|/c)\cos(\theta_0)\big)^{-1}\omega$. Moreover, the resultant energy absorbed by $\Sigma$ from the pencil is just the absorption coefficient $A(\omega)$ times this quantity. The total energy $\Delta E_2$ absorbed by $\Sigma$ in time $\Delta t$ is then obtained by integration and takes the form

$$\Delta E_2 = C \Phi_0(T_0) \Delta a \Delta t, \tag{3.7}$$

where

$$\Phi_0(T_0) = 2\gamma^{-3} \int_0^\infty d\omega \int_0^\pi d\theta_0 A(\omega) \omega^3 \sin^2(\theta_0) \times$$
$$\big(1 + (|v|/c)\cos(\theta_0)\big)^{-3} \Big[\exp\Big((\hbar\omega/\gamma kT)\big(1 + (|v|/c)\cos(\theta_0)\big)^{-1}\Big) - 1\Big]^{-1}. \tag{3.8}$$

It follows immediately from this formula that $\Phi_0$ is a continuous, monotonically increasing function of $T_0$ whose range is $[0, \infty)$.

We now infer from Eqs. (3.1), (3.3) and (3.7) that the net energy increment in the energy of $\Sigma$, relative to $K$, in time $\Delta t$ is

$$\Delta E = C[\Phi_0(T_0) - \Phi(T)] \Delta a \Delta t. \tag{3.9}$$

Hence, passing to the limit $\Delta t \to 0$, the rate of change of the energy $E$ of $\Sigma$ is

$$\frac{dE}{dt} = C[\Phi_0(T_0) - \Phi(T)] \Delta a. \tag{3.10}$$

**3.2. Evolution to the Equilibrium Temperature of $\Sigma$.** Since the functions $\Phi$ and $\Phi_0$ are continuous and monotonically increasing, with range $[0, \infty)$, it follows from

Eq. (3.10) that there is precisely one value, $\overline{T}$, of $T$ for which $E$ is stationary. Thus $\overline{T}$ is determined by the equation

$$\Phi(\overline{T}) = \Phi_0(T_0). \tag{3.11}$$

Moreover, since $\Phi_0$, as well as $\Phi$, increases monotonically and continuously with its argument, this formula implies that $\overline{T}$ is an increasing function of $T_0$.

In order to show that the temperature of $\Sigma$ evolves irreversibly to the value $\overline{T}$, we introduce the free energy function

$$F(E,V) = E - \overline{T}S(E,V) \tag{3.12}$$

and infer from Eqs. (2.3) and (3.10) that

$$\frac{d}{dt}F(E,V) = C[1 - \overline{T}/T][\Phi_0(T_0) - \Phi(T)]\Delta a$$

and consequently, by Eq. (3.11), that

$$\frac{d}{dt}F(E,V) = C[1 - \overline{T}/T][\Phi(\overline{T}) - \Phi(T)]\Delta a. \tag{3.13}$$

Since $\Phi$ is a continuous monotonically increasing function of temperature it follows immediately from this equation that $dF/dt$ is negative except at $T = \overline{T}$, where it is zero. This leads us to the following result.

(II) *$F$ serves as a Lyapounov function whose monotonic decrease with time ensures that the temperature of $\Sigma$ evolves irreversibly to a stable terminal value $\overline{T}$, which is the temperature of the moving system $\Sigma_0$, as registered by the thermometer fixed in $K$. Moreover, as noted following Eq. (3.9), this temperature is an increasing function of $T_0$.*

**3.3. Dependence of $\overline{T}$ on the Parameters of the Model.** We now remark that, by Eqs. (3.4) and (3.8), the functions $\Phi$ and $\Phi_0$ depend on the form of the absorption coefficient $A(\omega)$, and therefore, by Eq. (3.11), so too does the temperature $\overline{T}$. In order to establish that this dependence is non-trivial, we consider the case where $A(\omega)$ is unity when $\omega$ lies in a narrow interval $[f, f + \Delta f]$ and is otherwise zero. In this case, it follows from Eqs. (3.4), (3.8) and (3.11) that

$$[\exp(\hbar f/k\overline{T}) - 1]^{-1} = 2\pi^{-1}\gamma^{-3}\int_0^{\pi} d\theta_0 \sin^2(\theta_0)\times$$

$$\big(1 + (|v|/c)\cos(\theta_0)\big)^{-3}\big[\exp\big((\hbar f/\gamma kT_0)\big(1 + (|v|/c)\cos(\theta_0)\big)^{-1}\big) - 1\big]^{-1}. \tag{3.14}$$

In order to establish that $\overline{T}$ depends non-trivially on the frequency $f$, i.e. that it is not just a constant, we show $\overline{T}$ tends to different limits as $f$ tends to zero and infinity. Thus, in the case of small $f$, we may approximate the quantities in the square brackets on the

left and right hand sides of Eq. (3.14) by the exponents occurring there. Thus we find that

$$\overline{T}\to 2\pi^{-1}\gamma^{-2}T_0\int_0^{\pi}d\theta_0\sin^2(\theta_0)\big(1+(|v|/c)\cos(\theta_0)\big)^{-2}\ \text{as}\ f\to 0. \tag{3.15}$$

On the other hand, for large $f$, we may discount the terms $-1$ in the square brackets on both sides of Eq. (3.14), thereby obtaining the formula

$$\exp(-\hbar f/k\overline{T})=2\pi^{-1}\gamma^{-3}\times$$

$$\int_0^{\pi}d\theta_0\sin^2(\theta_0)\big(1+(|v|/c)\cos(\theta_0)\big)^{-3}\exp\big(-(\hbar f/\gamma kT_0)\big(1+(|v|/c)\cos(\theta_0)\big)^{-1}\big).$$

For large $f$, the r.h.s. of this equation is dominated by the exponential term occurring therein and its logarithm reduces to the maximum value of the exponent for $\theta_0\in[0,\pi]$. Hence, using Eq. (2.2), we find that

$$\overline{T}\to T_0\Big(\frac{1+|v|/c}{1-|v|/c}\Big)^{1/2}\ \text{as}\ f\to\infty. \tag{3.16}$$

This limit is evidently different from that of Eq. (3.15), since it follows easily from Eq. (2.2) that the latter limit is equal to $T_0\big(1+O(v^2/c^2)\big)$. Hence the temperature $\overline{T}$ must be a nontrivial, i.e. non-constant, function of $f$. It follows that this temperature depends on the parameter of the model and therefore we arrive at the following general conclusion.

(III) *According to the purely macroscopic picture, there is no intrinsic law of temperature transformations under Lorentz boosts.*

## 4. Conclusion

Our essential results are encapsulated by the assertions (I) of Sec. 2 and (II) and (III) of Sec. 3. The first of these is that, under the prescribed conditions, the composite of $\Sigma$ and $\Sigma_0$ cannot evolve to an equilibrium state, as given by the maximum entropy condition. However, as in the case of Sec. 3, where these systems interact via radiative transfer, their coupling can drive $\Sigma$ into an equilibrium state whose temperature $\overline{T}$ varies not only with $T_0$ but also with the parameters of the thermometer $\Sigma$. From this we conclude that in the purely macroscopic picture, as in the microstatistical one of [8]-[11], there is no intrinsic law of temperature transformation under Lorentz boosts.

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